Institut fr Theoretische Physik, Universitt Zürich

Loesung fuer Blatt 0 ,,Elektrodynamik“

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Aufgabe 1 Der total antisymmetrische Tensor ϵ_{ijk}

(a) Das Levi-Civita-Symbol lsst sich als Spatprodukt dreier orthogonaler Einheitsvektoren darstellen:

$$\epsilon_{ijk} = \det \begin{pmatrix} - & \hat{e}_i & - \\ - & \hat{e}_j & - \\ - & \hat{e}_k & - \end{pmatrix} =: \det A \tag{1}$$

$$\epsilon_{lmn} = \det \begin{pmatrix} - & \hat{e}_l & - \\ - & \hat{e}_m & - \\ - & \hat{e}_n & - \end{pmatrix} =: \det B \tag{2}$$

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmn} &= \det A \cdot \det B = \det A \cdot \det B^T = \det(AB^T) = \left| \begin{pmatrix} - & \hat{e}_i & - \\ - & \hat{e}_j & - \\ - & \hat{e}_k & - \end{pmatrix} \begin{pmatrix} - & - & - \\ \hat{e}_l & \hat{e}_m & \hat{e}_n \\ - & - & - \end{pmatrix} \right| \\ &= \begin{vmatrix} \hat{e}_i \cdot \hat{e}_l & \hat{e}_i \cdot \hat{e}_m & \hat{e}_i \cdot \hat{e}_n \\ \hat{e}_j \cdot \hat{e}_l & \hat{e}_j \cdot \hat{e}_m & \hat{e}_j \cdot \hat{e}_n \\ \hat{e}_k \cdot \hat{e}_l & \hat{e}_k \cdot \hat{e}_m & \hat{e}_k \cdot \hat{e}_n \end{vmatrix} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \end{aligned} \tag{3}$$

(Wikipedia)

(b) Mit $\delta_{ii} = 3, \quad \delta_{im}\delta_{ki} = \delta_{mk}$;

$$\begin{aligned} \epsilon_{ijk}\epsilon_{imn} &= \delta_{ii}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - \delta_{ki}\delta_{jm}\delta_{in} \\ &\quad -\delta_{km}\delta_{jn}\delta_{ii} - \delta_{kn}\delta_{ji}\delta_{im} \\ &= 3\delta_{jm}\delta_{kn} + \delta_{jm}\delta_{km} + \delta_{nj}\delta_{km} - \delta_{jm}\delta_{kn} - 3\delta_{km}\delta_{jn} - \delta_{kn}\delta_{jm} \\ &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \end{aligned} \tag{4}$$

$$\epsilon_{ijk}\epsilon_{ijn} = \delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj} = 3\delta_{kn} - \delta_{kn} = 2\delta_{kn} \tag{5}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{kk} = 2 \times 3 = 6 \tag{6}$$

(c) Mit $\vec{a} \times \vec{b} = \epsilon_{ijk}\vec{e}_i a_j b_k$;

$$\begin{aligned} [\vec{a} \times (\vec{b} \times \vec{c})]_i &= \epsilon_{ijk} a_j (\vec{b} \times \vec{c})_k = \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m \\ &= \epsilon_{kij}\epsilon_{klm} a_j c_m b_l = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m \\ &= (\vec{a} \cdot \vec{c}) b_i - (\vec{a} \cdot \vec{b})) c_i \end{aligned} \tag{7}$$

wobei in der letzten Gleichung $\delta_{jm} a_j c_m = a_j c_j = \vec{a} \cdot \vec{c}$ verwendet wird.

– bitte wenden –

Aufgabe 2 Der Nabla-Operator

(a) Normalenvektor an $f(\vec{x}) = \text{const.}$:

$$\vec{n} = \vec{\nabla} f(\vec{x}); \quad \vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \tag{8}$$

$$\tag{9}$$

Normiert:

$$\hat{\vec{n}} = \frac{1}{\left|\vec{\nabla} f(\vec{x})\right|} \vec{\nabla} f(\vec{x}) \tag{10}$$

Ellipsoid: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$,

$$\vec{n} = \begin{pmatrix} 2x_1/a^2 \\ 2x_2/b^2 \\ 2x_3/c^2 \end{pmatrix}; \quad \hat{\vec{n}} = \frac{1}{\sqrt{x_1^2/a^4 + x_2^2/b^4 + x_3^2/c^4}} \begin{pmatrix} x_1/a^2 \\ x_2/b^2 \\ x_3/c^2 \end{pmatrix} \tag{11}$$

zeigt $\hat{\vec{n}}$ nach aussen? Betr. $P(0, 0, |c|)$;

$$\hat{\vec{n}} P = |c| \begin{pmatrix} 0 \\ 0 \\ 1/|c| \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{12}$$

$\Rightarrow$ $\hat{\vec{n}}$ zeigt in P nach aussen $\Rightarrow$ $\hat{\vec{n}}$ zeigt überall nach aussen, da $\hat{\vec{n}}$ eine stetige Funktion von (x_1, x_2, x_3) ist und somit nicht die Orientierung relativ zur Fläche wechselt.

(b) $\vec{\nabla}\varphi = \begin{pmatrix} x_2 x_3 \\ x_1 x_3 \\ x_1 x_2 \end{pmatrix}; \quad \vec{a} \cdot \vec{\nabla}\varphi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} x_2 x_3 \\ x_1 x_3 \\ x_1 x_2 \end{pmatrix} = x_1 x_2 + x_1 x_3 + x_2 x_3$

(c) $r = \sqrt{x_1^2 + x_2^2 + x_3^2}; \quad \vec{\nabla} r$:

$$(\vec{\nabla} r)_i = \frac{\partial}{\partial x_i} \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{1}{2} \frac{2x_i}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{x_i}{r} \tag{13}$$

$$(\vec{\nabla} r)_i = \partial_i r = \frac{x_i}{r} \tag{14}$$

$$(\vec{\nabla} f(r))_i = \partial_i f(r) = \frac{\partial f(x)}{\partial r} \frac{x_i}{r} \tag{15}$$

$$\Rightarrow \vec{\nabla} r = \frac{\vec{r}}{r} = \hat{\vec{r}} \tag{16}$$

$$\left(\vec{\nabla} \frac{1}{r}\right)_i = -\frac{1}{2} \frac{2x_i}{\sqrt{x_1^2 + x_2^2 + x_3^2}^3} \quad \Rightarrow \quad \vec{\nabla} \frac{1}{r} = -\frac{\vec{r}}{r^3} \tag{17}$$

$$\vec{\nabla} f(r) = \frac{\partial f(r)}{\partial r} \cdot \vec{\nabla} r = \frac{\partial f(r)}{\partial r} \cdot \frac{\vec{r}}{r} \tag{18}$$

Aufgabe 3 Divergenz

(a)

$$\begin{aligned}
\vec{\nabla}\cdot\vec{x} &= 3 && (19)\\
\vec{\nabla}\cdot(r\vec{a}) &= \partial_i(a_i r) = a_i\partial_i r \overbrace{=}^{2c)} \frac{a_i x_i}{r} = \frac{\vec{a}\cdot\vec{r}}{r} && (20)\\
\vec{\nabla}\cdot(r^n\vec{x}) &= \partial_i(r^n x_i) = (\partial_i x_i)r^n + x_i\partial_i r^n\\
&= 3r^n + x_i n r^{n-1}\frac{x_i}{r} = 3r^n + nr^n\\
&= (3+n)r^n && (21)
\end{aligned}$$

(b)

$$\vec{\nabla}\cdot(\vec{B}\times\vec{x}) = \partial_i(\vec{B}\times\vec{x})_i = \partial_i\epsilon_{ijk}B_jx_k = \underbrace{\epsilon_{ijk}}_{antisymm}\, B_j\, \underbrace{\delta_{ik}}_{symm} = 0 \qquad (22)$$

(c) n=3:

$$\vec{\nabla}\cdot\left(\frac{\vec{x}}{r^3}\right)\underbrace{=}_{3a)}0 \qquad (23)$$

(d)

$$\begin{aligned}
\vec{\nabla}\cdot\left(\hat{\vec{x}}f(r)\right) &= \partial_i\left(x_i\frac{f(r)}{r}\right) = \frac{f(r)}{r}\partial_i x_i + x_i\partial_i\frac{f(r)}{r}\\
&= 3\frac{f(r)}{r} + x_i\left(\frac{\partial}{\partial r}\frac{f(r)}{r}\right)\frac{x_i}{r} = 3\frac{f(r)}{r} + r\frac{f'(r)r - f(r)}{r^2}\\
&= 2\frac{f(r)}{r} + \frac{\partial f(r)}{\partial r} && (24)
\end{aligned}$$

Aufgabe 4 Rotation

(a)

$$\begin{aligned}
[\vec{\nabla}\times(\vec{x}f(r))]_i &= \epsilon_{ijk}\partial_j[x_k f(r)] = \overbrace{\epsilon_{ijk}\underbrace{[\partial_j x_k]}_{\delta_{jk}}f(r)}^{0} + \epsilon_{ijk}x_k\partial_j f(r)\\
&= \epsilon_{ijk}x_k\left(\frac{\partial f(r)}{\partial r}\right)\frac{x_j}{r}\\
&= 0 \quad \text{(Symmetrie)} && (25)
\end{aligned}$$

(b)

$$\begin{aligned}[\vec{\nabla} \times (\vec{B} \times \vec{x})]_i &= \epsilon_{ijk}\partial_j(\vec{B} \times \vec{x})_k = \epsilon_{ijk}\partial_j \epsilon_{klm} B_l x_m \\ &= \epsilon_{ijk}\epsilon_{klm} B_l \delta_{jm} = \epsilon_{ijk}\epsilon_{klj} B_l = \epsilon_{jki}\epsilon_{jkl} B_l \\ &= 2\delta_{il} B_l = 2B_i \quad \Rightarrow \vec{\nabla} \times (\vec{B} \times \vec{x}) = 2\vec{B} \end{aligned} \tag{26}$$

(c)

$$[\vec{\nabla} \times \vec{\nabla}\phi]_i = \epsilon_{ijk}\partial_j\partial_k\phi = 0 \tag{27}$$

(d)

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \partial_i(\vec{\nabla} \times \vec{A})_i = \partial_i\epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}\partial_i\partial_j A_k = 0 \tag{28}$$

(e)

$$\begin{aligned}[\vec{\nabla} \times (\vec{A} \times \vec{B})]_i &= \epsilon_{ijk}\partial_j(\vec{A} \times \vec{B})_k = \epsilon_{ijk}\epsilon_{klm}\partial_j[A_l B_m] \\ &= \epsilon_{kij}\epsilon_{klm}\left((\partial_j A_l)B_m + A_l(\partial_j B_m)\right) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})[(B_m\partial_j A_l) + A_l(\partial_j B_m)] \\ &= B_j\partial_j A_i + A_i\partial_j B_j - B_i\partial_j A_j - A_j\partial_j B_i \\ &= A_i(\vec{\nabla} \cdot \vec{B}) - B_i(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})A_i - (\vec{A} \cdot \vec{\nabla})B_i \end{aligned} \tag{29}$$

$$\Rightarrow [\vec{\nabla} \times (\vec{A} \times \vec{B})] = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B}$$

Aufgabe 5 δ Funktion

(a)

$$\delta_n(x) = \frac{1}{\pi}\frac{n}{(1+n^2x^2)} \quad \lim_{n\to\infty}\frac{1}{\pi}\frac{n}{(1+n^2x^2)} = \lim_{n\to\infty}\frac{1}{\pi}\frac{\overbrace{1/n}^{0}}{\underbrace{1/n^2+x^2}_{x^2}} = 0 \quad (x \neq 0) \tag{30}$$

(b)

$$\delta_n(x) = \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2 \quad \lim_{n\to\infty}\frac{n}{\pi}\frac{\sin^2(nx)}{n^2x^2} = \lim_{n\to\infty}\frac{1}{n}\frac{\overbrace{\sin^2(nx)}^{[0,1]}}{\underbrace{nx^2}_{\infty}} = 0 \quad (x \neq 0) \tag{31}$$

(c)

$$\delta_n(x) = ne^{-\pi n^2x^2} \tag{32}$$

i)

$$\lim_{n\to\infty} e^{-\pi n^2x^2} = \lim_{n\to\infty}\frac{n}{e^{\pi n^2x^2}} \underbrace{=}_{l'H\hat{o}pital} \lim_{n\to\infty}\frac{1}{2n\pi x^2 e^{\pi n^2x^2}} = 0 \quad (x \neq 0) \tag{33}$$

ii)

$$\delta_n(x) \approx 0, \quad \text{sofern} \quad x \gg 1/(n\sqrt{\pi}) \quad \Rightarrow \text{für} \quad n \to \infty \quad \text{gilt} \quad \delta_n(x) = 0 \quad \forall x \neq 0 \tag{34}$$