Fermion propagator

$$\langle 0|T(\psi(x)\bar{\psi}(x'))|0\rangle = i S_F(x-x')$$

$$= i \lim_{\varepsilon\to 0_+} \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{\not p + m}{p^2-m^2+i\varepsilon}$$

Gauge transformation of photon field

$$A^\mu \to A^\mu + \partial^\mu \Lambda$$

Lorentz gauge: $\partial_\mu A^\mu = 0$ from $\Box\Lambda = -\partial_\mu A^\mu$

Coulomb gauge: $\frac{\partial \Lambda'}{\partial t} = -\phi \Rightarrow \phi = 0 \quad \vec{\nabla}\cdot\vec{A} = 0$

Gauge-fixed Lagrangian

$$\mathcal{L}_\lambda = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2$$

yields:

$$\Box A^\mu - (1-\lambda)\,\partial^\mu(\partial_\nu A^\nu) = 0$$

Feynman gauge

$$\lambda = 1 \Rightarrow \Box A^\mu = 0 \quad \text{and} \quad \pi^0 = -\partial_\mu A^\mu$$

Gupta - Bleuler quantization

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \sum_{\lambda=0}^{3} \left[\varepsilon_\mu^{(\lambda)}(k) a^{\lambda}(k) e^{-ikx} + \varepsilon_\mu^{(\lambda)}(k) a^{\lambda+}(k) e^{ikx} \right]$$

with: $k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}$: $\varepsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\varepsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\varepsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\varepsilon^{(1,2)}$ transversal
$\varepsilon^{(0)}$ scalar
$\varepsilon^{(3)}$ longitudinal

commutators $[A_\mu(\vec{x},t), \pi_\nu(\vec{x}',t)] = i g_{\mu\nu} \delta^3(\vec{x}-\vec{x}')$

$$[A_\mu(\vec{x},t), A_\nu(\vec{x}',t)] = 0$$

$$[\pi_\mu(\vec{x},t), \pi_\nu(\vec{x}',t)] = 0$$

$$\Rightarrow [a^{(\lambda)}(k), a^{(\lambda')+}(k')] = -g^{\lambda\lambda'}(2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

physical states defined by gauge condition: $\langle\Psi|\partial^\mu A_\mu|\Psi\rangle$

$$\Rightarrow \langle\Psi|a^{(0)+}(k)\, a^{(0)}(k)|\Psi\rangle = \langle\Psi|a^{(3)+}(k)\, a^{(3)}(k)|\Psi\rangle$$

Hamilton operator

$$\langle\Psi|:H:|\Psi\rangle = \langle\Psi| \int \frac{d^3k}{(2\pi)^3} k^0 \sum_{\lambda=1}^{2} a^{(\lambda)+}(k)\, a^{(\lambda)}(k) |\Psi\rangle$$

Fock space states of photon field

$|0\rangle$ Vacuum

$|1(\vec{k},\lambda)\rangle = \sqrt{2E_k}\, a^{(\lambda)\dagger}(k)\,|0\rangle$ one-photon state

multi-photon states:

$$\sqrt{2E_{k_i}}\, a^{\lambda_i\dagger}(k_i)\, |n_1(\vec{k}_1,\lambda_1);\dots;n_i(\vec{k}_i,\lambda_i);\dots\rangle$$
$$= \sqrt{n_i+1}\, |n_1(\vec{k}_1,\lambda_1);\dots;(n_i+1)(\vec{k}_i,\lambda_i);\dots\rangle$$

$$\frac{1}{\sqrt{2E_{k_i}}}\, a^{\lambda_i}(k_i)\, |n_1(\vec{k}_1,\lambda_1);\dots,n_i(\vec{k}_i,\lambda_i);\dots\rangle$$
$$= \sqrt{n_i}\, |n_1(\vec{k}_1,\lambda_1);\dots,(n_i-1)(\vec{k}_i,\lambda_i);\dots\rangle$$

Photon propagator

$$i D_{\mu\nu,F}(x-x') = \langle 0|T(A_\mu(x)A_\nu(x'))|0\rangle$$
$$= -g_{\mu\nu}\, i\Delta_F(x-x')$$
$$= i \lim_{\varepsilon\to 0_+} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2+i\varepsilon} (-g_{\mu\nu})$$

Interaction picture

$$H = H_0 + H_{int}$$

states: $\Psi_I = e^{iH_0 t}\,\Psi_S \quad \Rightarrow \quad i\frac{\partial \Psi_I}{\partial t} = H_{int,I}\,\Psi_I$

operators: $O_I = e^{iH_0 t}\, O_S\, e^{-iH_0 t} \quad \Rightarrow \quad i\frac{\partial O_I}{\partial t} = [O_I, H_0]$

Time evolution operator

$$\Psi_I(t) = U(t,t_0)\,\Psi_I(t_0)$$

with: $U(t,t_0) = T\left(\exp\left(-i\int_{t_0}^{t} dt'\, H_{int}(t')\right)\right)$

Vacuum expectation values

$$\langle \Omega | T(\phi_H(x)\,\phi_H(y)) | \Omega \rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0 | T(\phi_I(x)\,\phi_I(y) \exp(-i\int_{-T}^{T} dt\, H_I(t))) | 0 \rangle}{\langle 0 | T(\exp(-i\int_{-T}^{T} dt\, H_I(t))) | 0 \rangle}$$

Wick-Theorem

$$\begin{aligned}
T(ABC\ldots XYZ) &= :ABC\ldots XYZ: \\
&+ :\underbrace{AB}C\ldots XYZ: + \ldots + :ABC\ldots XYZ: \\
&+ :A\underbrace{BC}\ldots XYZ: + \ldots \\
&+ :\underbrace{AB}\,\underbrace{CD}\ldots XYZ: + \underbrace{ABCD}\ldots XYZ: \\
&+ \ldots
\end{aligned}$$

with $\langle 0 | T(\phi_A(x_1)\,\phi_B(x_2)) | 0 \rangle \equiv \underbrace{\phi_A(x_1)\,\phi_B(x_2)}$

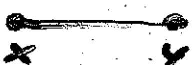
$\lambda\phi^4$ - theory

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_{int}}$$

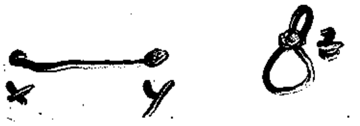
Two-point function

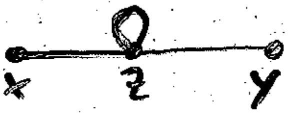
$$\langle \Omega | T(\phi_H(x)\phi_H(y)) | \Omega \rangle$$

$$= \langle 0 | T\left(\phi(x)\phi(y) + \phi(x)\phi(y)\left[-i\int dt H_I(t)\right] + \dots\right) | 0 \rangle$$

with $\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle = i D_F(x-y) =$ 

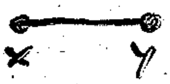
$$\langle 0 | T\left(\phi(x)\phi(y)\left(-\frac{i\lambda}{4!}\right)\int d^4z\, \phi(z)\phi(z)\phi(z)\phi(z)\right) | 0 \rangle$$

$= 3\left(-\frac{i\lambda}{4!}\right)\int d^4z$ ()

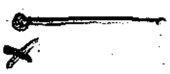
$+ 12\left(-\frac{i\lambda}{4!}\right)\int d^4z$ ()

Feynman rules (connected diagrams)

position space | momentum space

 $= iD_F(x-y)$ | propagator  $= \frac{i}{p^2 - m^2 + i\varepsilon}$

 $= -i\lambda\int d^4z$ | vertex  $= -i\lambda$

 $= 1$ | external state  $= -e^{-ipx}$

Symmetry factors

- $\frac{1}{2}$ for propagator connecting vertex with itself
- $\frac{1}{k!}$ for interchangability of k vertices or k propagators

S-Matrix

initial state $|\phi_i\rangle = \lim_{t \to -\infty} |\Psi(t)\rangle$

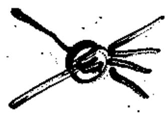
final state $\langle \phi_f| = \lim_{t \to +\infty} \langle \Psi(t)|$

$$S_{fi} = \langle \phi_f | S | \phi_i \rangle = \lim_{t \to +\infty} \langle \phi_f | \Psi(t) \rangle$$

$$S = U(-\infty, \infty)$$

$$S_{fi} = \delta_{fi} + i (2\pi)^4 \delta^4(P_f - P_i)\, T_{fi}$$

no interaction



interaction

Feynman rules of QED

$$\mathcal{L} = \mathcal{L}_0^{Dirac} + \mathcal{L}_0^{Photon} + \mathcal{L}_I \quad \text{with} \quad \mathcal{L}_I = -e \bar{\Psi} \gamma_\mu \Psi A^\mu$$

$\Psi^+(x)$ Electron absorption

$\bar{\Psi}^+(x)$ Positron absorption

$\bar{\Psi}^-(x)$ Electron emission

$\Psi^-(x)$ Positron emission

$A^+(x)$ Photon absorption

$A^-(x)$ Photon emission

Vertex

$-ie\, \bar{\Psi}(x) \gamma_\mu \Psi(x) A^\mu$

$= -ie\, \gamma_\mu \cdot$ (Vertex)

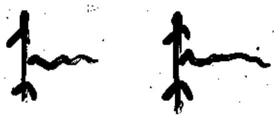
$\underbrace{\Psi(x_2) \bar{\Psi}(x_1)} = i S_F(x_2 - x_1)$ Fermion propagator

$\underbrace{A^\mu(x_2) A^\nu(x_1)} = i D_F^{\mu\nu}(x_2 - x_1)$ Photon propagator

S-operator to second order

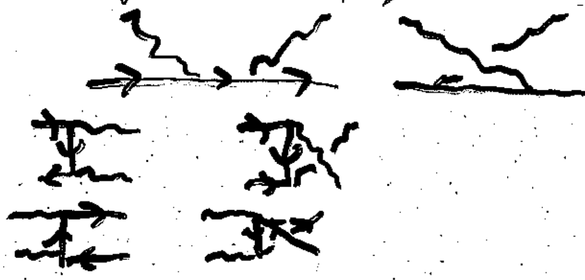
$$S^{(2)} = \frac{1}{2!}(-ie)^2 \int d^4x_1\, d^4x_2\; T\Big[\bar\Psi(x_1)\gamma_{\mu_1}\Psi(x_1)A^{\mu_1}(x_1)\; \bar\Psi(x_2)\gamma_{\mu_2}\Psi(x_2)A^{\mu_2}(x_2)\Big]$$

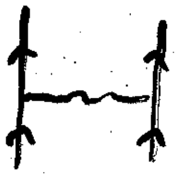
Contains:



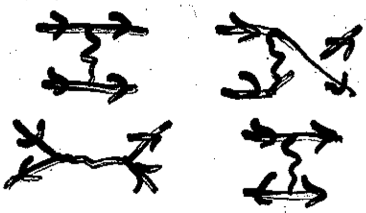
independent emission/absorption: unphysical

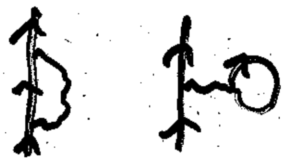
Compton scattering
Pair annihilation
Pair creation





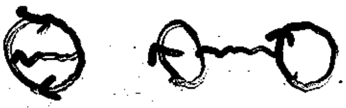
Møller scattering
Bhabha scattering





Electron self-energy

Photon self-energy (vacuum polarization

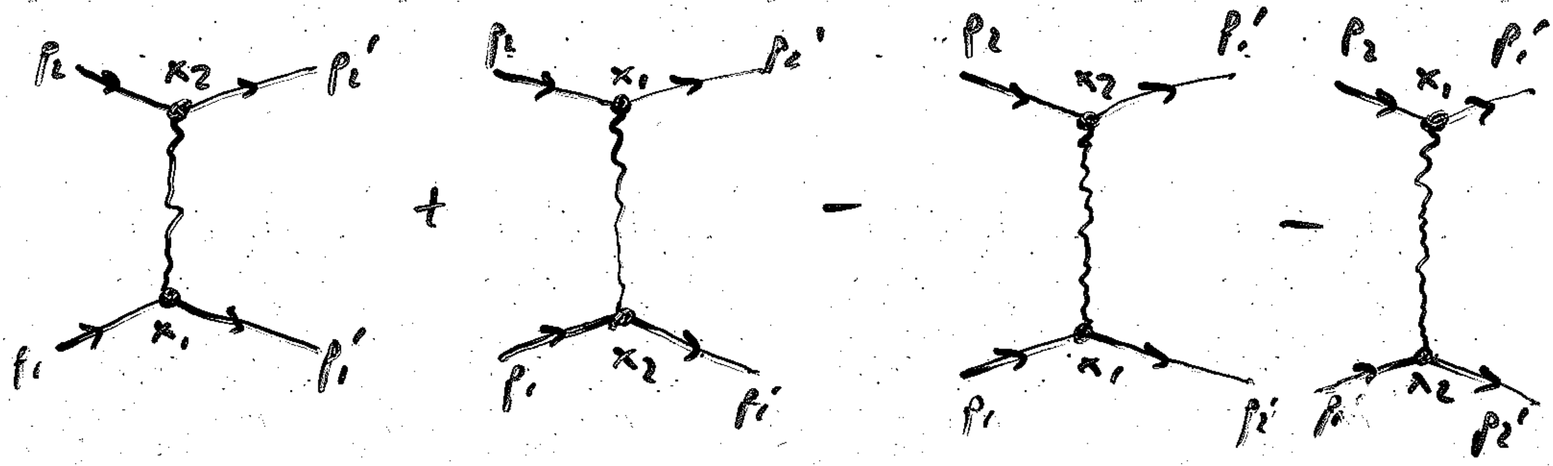


disconnected vacuum graphs

Møller scattering. $e^-e^- \to e^-e^-$

$$S_{fi} = \frac{(-ie)^2}{2!} \int d^4x_1\, d^4x_2 \sqrt{16 E_1 E_2 E_1' E_2'}$$

$$\langle 0 | a_{s_2'}(p_2')\, a_{s_1'}(p_1') | :\bar\Psi(x_1)\gamma^\mu \Psi(x_1)\bar\Psi(x_2)\gamma_\nu \Psi(x_2): \underbrace{A^\mu(x_1) A^\nu(x_2)}\, | a^+_{s_1}(p_1)\, a^+_{s_2}(p_2) |0\rangle$$



$$= (-ie)^2 (2\pi)^4\, \delta^4(p_1' + p_2' - p_1 - p_2)$$

$$\Big[\bar u_{s_2'}(p_2')\gamma_\mu u_{s_2}(p_2)\; iD_F^{\mu\nu}(p_1'-p_1)\; \bar u_{s_1'}(p_1')\gamma_\nu u_{s_1}(p_1)$$

$$-\bar u_{s_2'}(p_2')\gamma_\mu u_{s_1}(p_1)\; iD_F^{\mu\nu}(p_1'-p_2)\; \bar u_{s_1'}(p_1')\gamma_\nu u_{s_2}(p_2)\Big]$$

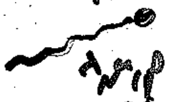
Feynman rules in momentum space

incoming electron  $u_s(p)$

outgoing electron  $\bar{u}_s(p)$

incoming positron  $\bar{v}_s(p)$

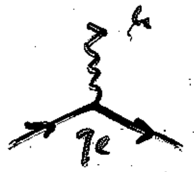
outgoing positron  $v_s(p)$

incoming photon  $\varepsilon^{\mu}_{\lambda}(p)$

outgoing photon  $\varepsilon^{\mu *}_{\lambda}(p)$

electron propagator  $\dfrac{i(\not{p}-m)}{p^2-m^2+i\varepsilon}$

photon propagator  $-\dfrac{i g^{\mu\nu}}{k^2+i\varepsilon}$

electron-photon-electron vertex  $-ieq_e\gamma^{\mu}$

Transition probability

$$w_{fi} = \frac{|S_{fi}|^2}{T}$$

$$dw_{fi} = \frac{V^{1-m}}{(2\pi)^{3n-4}}\,\delta^4(P_f-P_i)\,|M_{fi}|^2\prod_{i=1}^{m}\frac{1}{2E_i}\prod_{f=1}^{n}\frac{d^3p_f}{2E_f}$$

Decay width

$$\Gamma_{a\to n} = \frac{1}{\tau_{a\to n}} = \frac{1}{2E_a}(2\pi)^4\int\frac{d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{d^3p_n}{(2\pi)^3 2E_n}\,\delta^4(P_f-P_a)\,|M_{fa}|^2$$