QFT I
Exercise Sheet 3

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Exercise 1 [*Chirality and helicity*]

Show that for $m = 0$, chirality and helicity are equivalent.

Exercise 2 [*Angular momentum conservation*]

By explicit calculation, show that the total angular momentum

$$\vec{J} = \vec{x} \times \vec{p} + \frac{\hbar}{2}\vec{\Sigma}$$

and the Dirac-Hamilton-operator for a central potential

$$H = c\sum_{k=1}^{3} \gamma^0\gamma^k\, p^k + \gamma^0 mc^2 + e\,\Phi(|\vec{x}|)\,, \qquad \text{commute.}$$

Exercise 3 [*Gordon identity*]

a) Derive the Gordon identity that holds for massless particles:

$$\langle u^{\pm}|\gamma^\mu|u^{\pm}\rangle := \bar{u}_{\pm}(k)\gamma^\mu u_{\pm}(k) = 2k^\mu\ .$$

Here, we have introduced the spinor helicity notation which is very useful to obtain compact expressions for multi-particle scattering amplitudes.

b) Prove that the projection operator onto states with positive/negative helicity can be expressed as

$$|u^{\pm}\rangle\langle u^{\pm}| = \frac{1}{2}(1 \pm \gamma_5)\ \not{k}\ .$$

Hint: First, prove that the massless spinors can be written in this useful form ($c = 1$):

$$u_+(k) = v_-(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{k^+} \\ \sqrt{k^-} e^{i\varphi_k} \\ \sqrt{k^+} \\ \sqrt{k^-} e^{i\varphi_k} \end{pmatrix} , \; u_-(k) = v_+(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{k^-} e^{-i\varphi_k} \\ -\sqrt{k^+} \\ -\sqrt{k^-} e^{-i\varphi_k} \\ \sqrt{k^+} \end{pmatrix}$$

with

$$e^{\pm i\varphi_k} = \frac{k^1 \pm i k^2}{\sqrt{(k^1)^2 + (k^2)^2}} = \frac{k^1 \pm i k^2}{\sqrt{k^+ k^-}} \; , \; k^\pm = k^0 \pm k^3$$

Exercise 4 [*Maxwell's equations*]

Starting from the Lagrangian of the electromagnetic field,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu$$

derive the inhomogenous Maxwell's equations (the Euler-Lagrange equations of $\mathcal{L}$). The field strength tensor and the four-current are given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} , \qquad j^\mu = \begin{pmatrix} \rho \\ \vec{j} \end{pmatrix} .$$