QFT I
Exercise Sheet 4

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Exercise 1 [*Energy-Momentum Tensor*]

Calculate the energy-momentum tensor $T^{\mu\nu}$ for the free electromagnetic field with Lagrange density

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

defined as

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}.$$

Check that $\partial_\mu T^{\mu\nu} = 0$, then find a modified energy-momentum tensor which is symmetric, can be expressed as a function of $F^{\mu\nu}$ alone and for which we still have $\partial_\mu \tilde{T}^{\mu\nu} = 0$.

Calculate the components $\tilde{T}^{00}$ and $\tilde{T}^{0i}$ of the modified energy-momentum tensor.

Exercise 2 [*The Klein-Gordon Field as Harmonic Oscillators*]

Consider the Hamiltonian of a scalar field theory

$$H = \int \mathrm{d}^3 x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right], \quad \pi(\mathbf{x}) = \frac{\delta \mathcal{L}}{\delta(\partial_t \phi)}$$

Assume the following commutation relations for the field and the momentum density:

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= i \delta^3 (\mathbf{x} - \mathbf{y}), \\ [\pi(\mathbf{x}), \pi(\mathbf{y})] &= [\phi(\mathbf{x}), \phi(\mathbf{y})] = 0 \end{aligned} \tag{1}$$

show that we can rewrite this Hamiltonian as

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2} \left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}} \right] \right)$$

where the Fourier transforms are given by

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{px}} + a^\dagger_{\mathbf{p}} e^{-i\mathbf{px}} \right),$$

$$\pi(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{px}} - a^\dagger_{\mathbf{p}} e^{-i\mathbf{px}} \right).$$

We have abbreviated $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, our normalisations for the Fourier transform imply $\int \mathrm{d}^3x e^{i\mathbf{px}} = (2\pi)^3\delta(\mathbf{p})$. The commutation relations in (1) translate into $\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta(\mathbf{p} - \mathbf{p}')$.

Exercise 3 [*Causal Propagator*]

Show that $\Delta(x) = 0$ for a spacelike four-vector x ($x^2 < 0$), with

$$\Delta(x) = -i \int \frac{\mathrm{d}^4p}{(2\pi)^3} \delta(p^2 - m^2) e^{-ipx} \left(\Theta(p^0) - \Theta(-p^0)\right),$$

with $\Theta(a) = 1$ for $a \geq 0$ and $\Theta(a) = 0$ for $a < 0$.

Exercise 4 [*Supersymmetry*]

The Hamilton operator of the one-dimensional harmonic oscillator is

$$H_B = \frac{1}{2}\hbar\omega(a^\dagger a + a a^\dagger)$$

where the creation and the annihilation operator $a^\dagger$, a obey the commutation relations

$$[a, a^\dagger] = 1, \qquad [a, a] = [a^\dagger, a^\dagger] = 0.$$

Define a similar system with Hamiltonian

$$H_F = \frac{1}{2}\hbar\omega \left(b^\dagger b - b b^\dagger\right)$$

where we have the anticommutation relations ($\{A, B\} = AB + BA$)

$$\{b, b^\dagger\} = 1, \qquad \{b, b\} = \{b^\dagger, b^\dagger\} = 0.$$

Determine the spectrum of H_F. Are there ladder operators in analogy to the usual harmonic oscillator?

Consider the Hamiltonian $H = H_B \otimes 1 + 1 \otimes H_F$ operating on the tensor product of the two Hilbert spaces. Show that

$$Q = a^\dagger \otimes b, \qquad Q^\dagger = a \otimes b^\dagger$$

commute with H. What does this imply for the spectrum of H? Which states are degenerate? Show that

$$\{Q, Q^\dagger\} = \frac{1}{\hbar\omega} H.$$