QFT I
Exercise Sheet 5

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Exercise 1 [*Bilinear covariants*]
Given an arbitrary 4×4 constant matrix Γ, the bilinear $\bar{\psi}\Gamma\psi$ can be decomposed into terms that have definite transformation properties under the Lorenz group. This is achieved by writing Γ in terms of the following basis of sixteen 4×4 matrices:

$$\begin{aligned}
\Gamma^S &= \mathbb{1} \\
\Gamma^V_\mu &= \gamma_\mu \\
\Gamma^T_{\mu\nu} &= \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \\
\Gamma^P &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \\
\Gamma^A_\mu &= \gamma^5\gamma_\mu .
\end{aligned}$$

Show that:

(i) $(\Gamma^n)^2 = \pm\mathbb{1}$, for all n.

(ii) For every $n \neq S$ there is m such that

$$\Gamma^n\Gamma^m = -\Gamma^m\Gamma^n .$$

Prove that this means that $\mathrm{Tr}\,(\Gamma^n) = 0$ for $n \neq S$.

(iii) For every $a \neq b$ there is $n \neq S$ such that $\Gamma^a\Gamma^b = \pm\Gamma^n$

(iv) The Γ^n matrices are linearly independent.

Exercise 2 [*Energy-momentum tensor of the Dirac field*]
Given the Lagrange density for the Dirac field

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi,$$

calculate the energy momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi - g^{\mu\nu} \mathcal{L},$$

and show that

$$\begin{aligned} H = T^{00} &= \int d^3x \bar{\psi}(x) \left(-i\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m\gamma^0\right) \psi(x), \\ P^i = T^{0i} &= -i \int d^3x \bar{\psi}(x) \gamma^0 \partial^i \psi(x). \end{aligned}$$

Then use the Fourier expansion of the field operators

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right) \\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x} \right) \end{aligned}$$

to show that

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} \sum_s p^\mu \left(a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right).$$

Exercise 3 *Commutation of P^μ and Q*

Show that the four-momentum operator obtained in exercise 2 commutes with the charge operator

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} - b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right).$$